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Approximation with Kernels of Finite Oscillations I. Convergence

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1. INTRODUCTION

Let $C[a, b]$ denote the set of continuous (and bounded if $(a, b) = (-\infty, \infty)$) functions defined on $[a, b]$. Let $\mathscr{L}$ be a linear operator which maps $C[a, b]$ into itself and denote the transform of $f \in C[a, b]$ by $\mathscr{L}(f; x) \in C[a, b]$.

An important subclass of such operators is the class of positive linear operators. This class of operators has, in the past, received considerable attention. In particular, they have been widely investigated in regards to (1) the convergence of sequences of approximating functions $\{\mathscr{L}_n(f; x)\}$ to $f(x)$ in the Tchebycheff norm and (2) the determination of the degree of convergence for such sequences.

Korovkin, [3], investigates positive linear operators from both these standpoints. In connection with the former he presents some surprisingly elegant necessary and sufficient conditions for a sequence of positive linear operators to converge uniformly to $f(x) \in C[a, b]$. For the question of degree of convergence, he considers sequences of positive operators $\{\mathscr{T}_n(f; x)\}$ where $\mathscr{T}_n(f; x)$ is a trigonometric polynomial of degree n and f is periodic. He shows that for such sequences, the degree of convergence to f is not better than $O(n^{-2})$ (except possibly for some trivial functions) no matter how *smooth* f is.

In a more specialized framework, P. Butzer, [1], considers sequences of positive linear operators $\{\mathscr{H}_n\}$ which have the representation

$$\mathscr{H}_n(f; x) = \int_{-\infty}^{\infty} f(u)\, H(n(u - x))\, du = \int_{-\infty}^{\infty} f(u + x)\, H(nu)\, du. \quad (1.1)$$

* This paper is, in part, extracted from the author's Ph.D. thesis [2], written under the direction of Prof. John R. Rice.

The *generating kernel* $H(u)$ is a *positive* even function, continuous at $u = 0$ and absolutely integrable with

$$\int_{-\infty}^{\infty} H(u)\, du = 1.$$

In determining the degree of convergence of the operators $\mathscr{H}_n$, Butzer shows that if $f(x)$ is bounded and absolutely integrable on $(-\infty, \infty)$, then the following asymptotic expansion for the difference $\mathscr{H}_n(f; x) - f(x)$ holds for each x, where $f^{(2k)}$ exists and is $\neq 0$:

$$\lim_{n\to\infty} n^{2k}\left[\mathscr{H}_n(f; x) - f(x) - \sum_{i=1}^{k-1} \frac{M_{2i} f^{(2i)}(x)}{(2i)!\, n^{2i}}\right] = \frac{M_{2k} f^{(2k)}(x)}{(2k)!}. \tag{1.2}$$

The constants M_{2i} are the even moments of $H(t)$:

$$M_{2i} = \int_{-\infty}^{\infty} t^{2i} H(t)\, dt,$$

for $i = 1, 2,..., 2k$. We see that for the case $k = 1$, $\mathscr{H}_n(f; x) - f(x)$ is exactly of order $O(n^{-2})$ if $f''(x) \neq 0$.

Hence, for the positive linear operators investigated by Butzer, as well as as the positive trigonometric polynomial operators considered by Korovkin, the degree of approximation cannot be improved for any interesting classes of functions. Thus, if a better degree of approximation is to be achieved with either type of operator, the condition of positivity must be removed. This fact, then, raises the question of how an effective construction of a sequence $\{\mathscr{T}_n\}$ of *nonpositive* trigonometric polynomial operators of degree n or a sequence $\{\mathscr{H}_n\}$ of *nonpositive* operators of the form (1.1) can be produced, which assures a degree of convergence better than n^{-2} for large classes of functions. For example, if we let $M_2 = 0$ in (1.2), then with $k = 2$ and $f^{(4)}(x) \neq 0$, the difference $\mathscr{H}_n(f; x) - f(x)$ has exact order $O(n^{-4})$; and in general, if $M_{2i} = 0$, $i = 1,..., k - 1$, then $\mathscr{H}_n(f; x) - f(x)$ has exact order $O(n^{-2k})$. But in order that the moments M_{2i} be zero, $H(u)$ must become negative, and hence define nonpositive operators.

With this motivation in mind, we consider, in general, linear operators of the form

$$\mathscr{L}_n(f; x) = \int_{-a}^{a} f(t)\, K_n(t - x)\, dt, \qquad 0 < a \leqslant \infty,$$

where f and K_n are continuous and $2a$-periodic (or bounded, if $a = \infty$) functions and K_n is symmetric. We study the effect of allowing $K_n(u)$ to become negative in a prescribed manner, i.e., to oscillate finitely many times across the u axis. The definition of such nonpositive kernels and corresponding nonpositive operators $\mathscr{L}_n$ is made precise in Section 2.

The investigation is presented in two parts. This paper (*Part* I—*Convergence*) is concerned with the determination of necessary and sufficient conditions for the sequence of approximating functions $\{\mathscr{L}_n(f; x)\}$ to converge uniformly to $f(x)$. A subsequent paper (*Part* II—*Degree of Approximation*) deals with the question of degree of convergence in the case when $\mathscr{L}_n(f; x)$ is a trigonometric polynomial of degree n and when the operators $\mathscr{L}_n$ are defined by *kernels* $K_n(t)$ generated by one kernel function $H(t)$ thus: $K_n(t) = nH(nt)$. A method for constructing nonpositive operators is also discussed in Part II and some examples are given.

2. Definitions

For $0 < a \leqslant \infty$, let C_a denote the set of continuous, $2a$-periodic (or bounded, if $a = \infty$) functions defined on the real line. For $f \in C_a$, $\|f\|$ denotes the Tchebycheff norm of f, i.e., $\|f\| = \text{lub}\,|f(x)|$.

A number α is called a *simple zero* of a function $f \in C_a$ if $f(\alpha) = 0$ and if for some $\epsilon > 0$, $\alpha - \epsilon < \zeta_1 < \alpha < \zeta_2 < \alpha + \epsilon$ implies $f(\zeta_1)f(\zeta_2) \neq 0$, $\text{sgn}[f(\zeta_1)] = -\text{sgn}[f(\zeta_2)]$. If f has exactly k simple zeros in $(0, a)$, then α_i denotes the i-th zero, i.e., $0 < \alpha_1 < \cdots < \alpha_k$ are the k simple zeros of f. Similarly, if $f_n \in C_a$ and if f_n has exactly simple zeros in (0, a), then α_{ni}, $i = 1,..., k$, denote these zeros in their natural order.

Let $\mu(t)$ be an analytic, even function defined on $[-a, a]$ such that $\mu(0) = 0$ and $\mu(t)$ is strictly increasing on $[0, a]$. We denote the j-th μ-moment of $f \in C_a$ by $M_j(\mu, f)$, i.e.,

$$M_j(\mu, f) = \int_{-a}^{a} \mu^j(t) f(t)\, dt, \qquad j = 0, 1,...$$

and set $M_0(f) = M_0(\mu, f)$ whenever convenient (if $a = \infty$, the above integral might not exist for some $f \in C_a$). We note that two important functions which satisfy the definition of $\mu(t)$ are t^2 and $\sin^2 t/2$.

Definition 1. A function $K(t) \in C_a$, $0 < a \leqslant \infty$, is called a *kernel* if

(i) $K(t) = K(-t)$ (symmetry),

(ii) $M_0(K) = 1$ (normalization).

If, in addition, $K(t)$ has *exactly* k simple zeros α_i, $i = 1, 2,..., k$, in $(0, a)$ and for some μ,

(iii) $M_j(\mu, K) = 0, j = 1, 2,..., k$,

then $K(t)$ is called a *$2k$-zero kernel* with respect μ and is denoted by $K^{(k)}(t)$.

DEFINITION 2. A sequence of $2k$-zero kernels, $\{K_n^{(k)}\}$, is said to *peak* if $\lim_{n\to\infty} \alpha_{nk} = 0$.

Let $\mathscr{L} : C_a \to C_a$ denote the linear operator

$$\mathscr{L}(f; x) = \int_{-a}^{a} f(u)\, K(u - x)\, du, \qquad f \in C_a,$$

where K is a kernel. We note that since $f, K \in C_a$, $\mathscr{L}(f; x)$ may be written in the form

$$\mathscr{L}(f; x) = \int_{-a}^{a} f(x + u)\, K(u)\, du.$$

DEFINITION 3. If $K^{(k)}(t)$ is a $2k$-zero kernel, then the corresponding operator $\mathscr{L}^{(k)}$ is called a *$2k$-zero operator*. If a sequence $\{K_n^{(k)}\}$ of $2k$-zero kernels peaks, the corresponding sequence $\{\mathscr{L}_n^{(k)}\}$ is said to peak.

When we speak of the convergence of a sequence of operators $\{\mathscr{L}_n^{(k)}\}$, we mean the convergence of sequences $\{\mathscr{L}_n^{(k)}(f; x)\}$, $f \in C_a$.

3. CONVERGENCE OF $2k$-ZERO OPERATORS

We now consider the question of convergence of a sequence of $2k$-zero operators, $\{\mathscr{L}_n^{(k)}\}$, for all $f \in C_a$, $0 < a \leqslant \infty$. Specifically, in Theorem 1a we give sufficient conditions for $\{\mathscr{L}_n^{(k)}(f; x)\}$ to converge uniformly to $f(x)$. Theorem 1b is a partial converse to Theorem 1a.

These results, however, do not give any definitive information concerning the structure of the associated kernels, $K_n^{(k)}$, themselves, except in a rather general sense. Theorems 2 and 3 yield sufficient conditions for convergence in terms of more specific properties of the kernels $K_n^{(k)}$. In particular, we show that uniform convergence is assured for any sequence of 2-zero operators $\{\mathscr{L}_n^{(1)}\}$ if the sequence of associated kernels, $\{K_n^{(1)}\}$, peaks and if $K_n^{(1)}(t)$ satisfies a monotonicity condition for each n. For sequences $\{\mathscr{L}_n^{(k)}\}$, $k \geqslant 2$, uniform convergence is assured if the associated kernels, $K_n^{(k)}$, meet the above two requirements, and in addition, their zeros satisfy certain asymptotic separation conditions.

Preliminary Results

We first state without proof some basic facts about kernels (see [2]):

(1) $K^{(k)}(0) \geqslant 0$, $k \geqslant 0$.

(2) $K^{(k)}(t)$ has exactly $2k$ simple zeros, $\pm\alpha_i$, $i = 1, 2,\ldots, k$, in $(-a, a)$.

(3) If K is a kernel and satisfies condition (iii) of Definition 1, then K must have at least $2k$ simple zeros in $(-a, a)$. (Note that in the definition of a $2k$-zero kernel we assume (iii) holds and $K^{(k)}(t)$ has *exactly* k simple zeros in $(0, a)$.)

(4) If the sequence $\{K_n^{(k)}\}$ peaks, then for every $\delta > 0$,

$$\lim_{n\to\infty} \int_{|t|\geqslant\delta} | K_n^{(k)}(t)| \, dt = 0, \qquad k \geqslant 0.$$

The following two facts concerning functions $\mu(t)$ as in Section 2 are also needed.

(5) Let $\{x_n\}$ and $\{y_n\}$ be null sequences of positive numbers and let $\mu(t)$ be as in Section 2. Then, for some integer $r \geqslant 1$,

$$\overline{\lim_{n\to\infty}} \frac{\mu(y_n)}{\mu(x_n)} = \left[\overline{\lim_{n\to\infty}} \left(\frac{y_n}{x_n}\right)\right]^r.$$

In fact, if $\{y_n/x_n\}$ is bounded, then for large n,

$$\frac{\mu(y_n)}{\mu(x_n)} = \frac{\sum_{i=r}^{\infty} c_i y_n^{\,i}}{\sum_{i=r}^{\infty} c_i x_n^{\,i}} = \frac{c_r \left(\dfrac{y_n}{x_n}\right)^r + \left(\dfrac{y_n}{x_n}\right)^r \sum_{i=r+1}^{\infty} c_i y_n^{i-r}}{c_r + \sum_{i=r+1}^{\infty} c_i x_n^{i-r}}, \qquad r \geqslant 1, \quad c_r \neq 0,$$

from which the asserted equality follows. A similar argument proves it if $\{y_n/x_n\}$ is unbounded.

(6) Let $\{x_n\}$ and $\{y_n\}$ be null sequences with $0 < y_n < x_n$ and let $\mu(t)$ be as in Section 2. Then

$$\overline{\lim_{n\to\infty}} \frac{y_n}{x_n} < 1 \quad \text{implies} \quad \overline{\lim_{n\to\infty}} \left[\int_{y_n}^{x_n} \mu(t)\, dt\right] \Big/ [(x_n - y_n)\, \mu(x_n)] < 1.$$

We have, for some $c_r \neq 0$, $r \geqslant 1$, and for all large n,

$$\frac{\int_{y_n}^{x_n} \mu(t)\, dt}{(x_n - y_n)\, \mu(x_n)} = \frac{\sum_{i=r}^{\infty} \dfrac{c_i}{i+1} (x_n^{i+1} - y_n^{i+1})}{(x_n - y_n) \sum_{i=r}^{\infty} c_i x_n^{\,i}}$$

$$= \frac{\dfrac{c_r}{r+1}\left(1 + \dfrac{y_n}{x_n} + \cdots + \left(\dfrac{y_n}{x_n}\right)^r\right) + \sum_{i=r+1}^{\infty} \dfrac{c_i x_n^{i-r}}{i+1}\left(1 + \dfrac{y_n}{x_n} + \cdots + \left(\dfrac{y_n}{x_n}\right)^i\right)}{c_r + \sum_{i=r+1}^{\infty} c_i x_n^{i-r}}.$$

The two last infinite sums converge to 0 as $n \to \infty$, and hence

$$\overline{\lim_{n\to\infty}} \frac{\int_{y_n}^{x_n} \mu(t)\,dt}{(x_n - y_n)\,\mu(x_n)} = \overline{\lim_{n\to\infty}} \frac{1}{r+1}\left(1 + \frac{y_n}{x_n} + \cdots + \left(\frac{y_n}{x_n}\right)^r\right) < 1.$$

We now may state:

THEOREM 1a. *If* $\{K_n^{(k)}\}$ *peaks and* $\{M_0(|\, K_n^{(k)}\,|)\}$ *is bounded, then* $\{\mathscr{L}_n^{(k)}(f; x)\}$ *converges uniformly to* $f(x) \in C_a$ *as* $n \to \infty$.

A partial converse to this theorem is:

THEOREM 1b. *If* $\{\mathscr{L}_n^{(k)}(f; x)\}$ *converges uniformly to* $f(x) \in C_a$, then $\{M_0(|\, K_n^{(k)}\,|)\}$ *is bounded.*

The proofs of these theorems are omitted (see [2]). We note only that Theorem 1a follows from Fact 4, and Theorem 1b follows from the Uniform Boundedness Principle and the fact that for the norm of the operators $\mathscr{L}_n^{(k)}$ we have $\|\, \mathscr{L}_n^{(k)} \,\| = M_0(|\, K_n^{(k)}\,|)$.

2-Zero Operators

Now let us examine more closely the relationship between the uniform convergence of a sequence of operators $\{\mathscr{L}_n^{(k)}\}$ and the behavior of the sequences of zeros $\pm\alpha_{ni}$, $i = 1,\ldots, k$, of the associated kernels $K_n^{(k)}(t)$.

First, consider the case when $k = 1$ and define

$$R_n = \int_{\alpha_{n1}}^{a} |\, K_n^{(1)}(t)|\, dt. \tag{3.1}$$

Since the function $\mu(t)$ corresponding to the $K_n^{(1)}$ is increasing on $[0, a]$, by the first mean value theorem of the integral calculus there is a $\xi_n \in (0, \alpha_{n1})$ and a $\zeta_n \in (\alpha_{n1}, a)$ such that

$$\int_0^{\alpha_{n1}} \mu(t)\, K_n^{(1)}(t)\, dt = \mu(\xi_n)(R_n + \tfrac{1}{2}) \quad \text{and} \quad \int_{\alpha_{n1}}^{a} \mu(t)\,|\, K_n^{(1)}(t)|\, dt = \mu(\zeta_n)\, R_n .$$

We obtain as an immediate corollary to Theorems 1a and 1b:

COROLLARY 1. *Suppose the sequence* $\{\mathscr{L}_n^{(1)}\}$ *peaks. Then a necessary and sufficient condition that* $\{\mathscr{L}_n^{(1)}(f; x)\}$ *converge uniformly to* $f(x)$ *is that* $\mu(\xi_n) = O(\mu(\zeta_n) - \mu(\xi_n))$.

Proof. We have

$$0 = \int_0^a \mu(t)\, K_n^{(1)}(t)\, dt = \mu(\xi_n)(R_n + \tfrac{1}{2}) - \mu(\zeta_n)\, R_n$$

and hence $R_n = \frac{1}{2}\mu(\xi_n)/(\mu(\zeta_n) - \mu(\xi_n))$. Therefore, R_n is uniformly bounded if, and only if, $\mu(\xi_n) = O(\mu(\zeta_n) - \mu(\xi_n))$, and the corollary follows from Theorems 1a and 1b.

A more descriptive sufficient condition for the uniform convergence of 2-zero operators, is given by the following

THEOREM 2. *Let $\{\mathscr{L}_n^{(1)}\}$ be a sequence of 2-zero operators which peaks. If the associated kernels $K_n^{(1)}(t)$ decrease on $[0, \alpha_{n1}]$ for each n, then $\{\mathscr{L}_n^{(1)}(f; x)\}$ converges uniformly to $f(x) \in C_a$.*

Proof. Let R_n be defined by (3.1) and $h_n > 0$ by

$$h_n \alpha_{n1} = \int_0^{\alpha_{n1}} K_n^{(1)}(t)\, dt = R_n + \tfrac{1}{2}. \tag{3.2}$$

There is a ν_n such that $h_n = K_n^{(1)}(\nu_n)$. The situation is illustrated in Fig. 1.

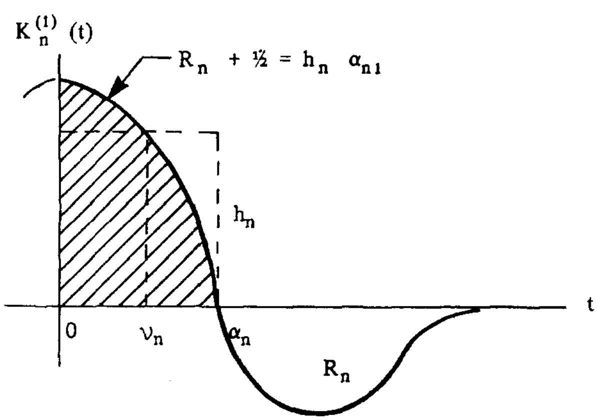

FIG. 1. 2-zero kernels.

We establish the inequality

$$\int_0^{\alpha_{n1}} \mu(t)\, K_n^{(1)}(t)\, dt \leqslant h_n \int_0^{\alpha_{n1}} \mu(t)\, dt. \tag{3.3}$$

From (3.2) we have

$$\int_0^{\nu_n} [K_n^{(1)}(t) - h_n]\,dt = \int_{\nu_n}^{\alpha_{n1}} [h_n - K_n^{(1)}(t)]\,dt.$$

Hence,

$$\begin{aligned}\int_0^{\alpha_{n1}} &\mu(t)[K_n^{(1)}(t) - h_n]\,dt \\ &\leqslant \mu(\nu_n) \int_0^{\nu_n} [K_n^{(1)}(t) - h_n]\,dt - \mu(\nu_n) \int_{\nu_n}^{\alpha_{n1}} [h_n - K_n^{(1)}(t)]\,dt = 0\end{aligned}$$

which establishes (3.3). From inequality (3.3) we then obtain

$$\begin{aligned}\int_0^{a} \mu(t)\,K_n^{(1)}(t)\,dt &= \int_0^{\alpha_{n1}} \mu(t)\,K_n^{(1)}(t)\,dt - \int_{\alpha_{n1}}^{a} \mu(t) \mid K_n^{(1)}(t)\mid dt \\ &< h_n \int_0^{\alpha_{n1}} \mu(t) - \mu(\alpha_{n1}) \int_{\alpha_{n1}}^{a} \mid K_n^{(1)}(t)\mid dt. \qquad (3.4)\end{aligned}$$

Now let c_n be such that

$$\int_0^{\alpha_n} \mu(t)\,dt = c_n \alpha_{n1} \mu(\alpha_{n1}). \qquad (3.5)$$

By Fact 6 there exists a $\delta < 1$ such that $c_n \leqslant \delta$ for all n. Therefore, from (3.2), (3.4), and (3.5) we have

$$\begin{aligned}0 = \tfrac{1}{2} M_1(\mu, K_n^{(1)}) &= \int_0^{a} \mu(t)\,K_n^{(1)}(t)\,dt \\ &< c_n \mu(\alpha_{n1})(R_n + \tfrac{1}{2}) - \mu(\alpha_{n1})\,R_n \\ &= -\mu(\alpha_{n1})[(1 - c_n)\,R_n - \tfrac{1}{2} c_n].\end{aligned}$$

Hence, since $\mu(\alpha_{n1}) > 0$, we must have $R_n < c_n/[2(1 - c_n)]$. But this implies

$$M_0(\mid K_n^{(1)} \mid) = 4R_n + 1 < \frac{1 + c_n}{1 - c_n} \leqslant \frac{1 + \delta}{1 - \delta}.$$

The theorem follows now immediately from Theorem 1a.

Clearly, the hypothesis that $K_n^{(1)}(t)$ is decreasing on $[0, \alpha_{n1}]$ is stronger than necessary for the conclusion of Theorem 2 to hold. A weaker, although less direct, property can replace the monotonicity condition. This property is derived from our method of proof in Theorem 2. Let R_n, c_n, and δ be as in the proof of Theorem 2 and let $K_n^{(1)}(t)$ be such that there is an $\eta > 0$, a nonnegative number $\rho < 1/\delta$, and for each n, an $h_n > 0$ such that $h_n \alpha_{n1} < \rho R_n + \eta$ and

$$\int_0^{\alpha_{n1}} \mu(t)\, K_n^{(1)}(t)\, dt < h_n \int_0^{\alpha_{n1}} \mu(t)\, dt.$$

Then we see that an inequality similar to (3.4) holds, so that

$$0 = \tfrac{1}{2} M_1(\mu, K_n^{(1)}) = \int_0^{\alpha_{n1}} \mu(t)\, K_n^{(1)}(t)\, dt - \int_{\alpha_{n1}}^{a} \mu(t) \mid K_n^{(1)}(t) \mid dt$$
$$< -\mu(\alpha_{n1})[(1 - c_n \rho)\, R_n - c_n \eta].$$

Therefore, $R_n < \eta\delta/(1 - \delta\rho)$, and hence $\{M_0(\mid K_n^{(1)} \mid)$ is uniformly bounded.

On the other hand, it is easy to give an example of a sequence of 2-zero operators which peaks, but whose associated kernels do *not* decrease on $[0, \alpha_{n1}]$ and $\{M_0(\mid K_n^{(1)} \mid)\}$ is unbounded. Let $\mu(t) = t^2$ and let

$$0 < u_n < v_n < w_n < a.$$

Then define $K_n^{(1)}(t)$ as the polygonal function shown in Fig. 2.

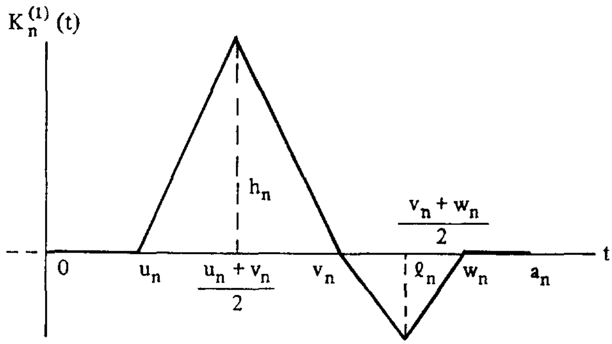

FIG. 2. Construction of 2-zero kernels with $\{M_0(\mid K_n^{(1)} \mid)\}$ unbounded.

For $K_n^{(1)}$ to be a 2-zero kernel, it must satisfy the conditions $M_0(K_n^{(1)}) = 1$ and $M_1(t^2, K_n^{(1)}) = 0$. This yields two linear equations in the unknowns h_n and ℓ_n. Solving for h_n, we obtain (see [2])

$$h_n = \frac{7w_n{}^2 + 10w_n v_n + 7v_n{}^2}{(v_n - u_n)(w_n - u_n)(7w_n + 7u_n + 10v_n)}.$$

Now set $u_n = o(1)$ and $w_n - u_n = o(u_n)$. Then $\{K_n^{(1)}\}$ peaks, and

$$\int_0^{v_n} K_n^{(1)}(t)\,dt = \tfrac{1}{2}(v_n - u_n)\,h_n \sim \frac{u_n}{w_n - u_n} \to \infty \qquad \text{as} \quad n \to \infty.$$

But this implies $\{M_0(|\,K_n^{(1)}\,|)\}$ is unbounded. By Theorem 1b, the sequence of approximating functions $\{\mathscr{L}_n^{(1)}(f;x)\}$ defined by the kernels $K_n^{(1)}(t)$ fails to converge uniformly to f, for some $f \in C_a$.

2k-Zero Operators, $k \geqslant 2$

We now direct our attention to $2k$-zero operators when $k \geqslant 2$. We first show, by example, that conditions as in Theorem 2 are not enough to assure uniform convergence to f when $k \geqslant 2$. Let $k = 2$, $\mu(t) = t^2$, and $0 < u_n < v_n < w_n < a$. Let $K_n^{(2)}(t)$ be as shown in Fig. 3.

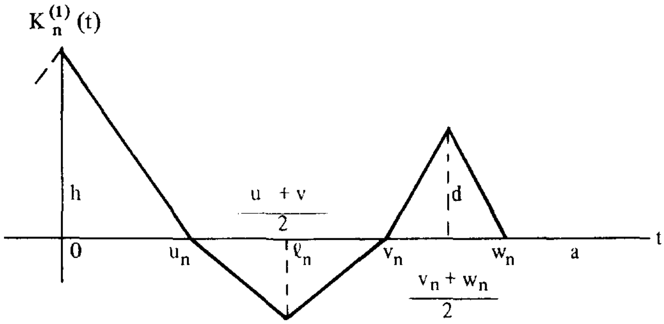

FIG. 3. Construction of 4-zero kernels with $\{M_0(|\,K_n^{(2)}\,|)\}$ unbounded.

For $K_n^{(2)}(t)$ to be a 4-zero kernel, it must satisfy $M_0(K_n^{(2)}) = 1$ and $M_j(t^2, K_n^{(2)}) = 0, j = 1, 2$. This yields three linear equations in the unknowns h_n, ℓ_n, and d_n:

$$\begin{aligned}
&\left[\frac{1}{u_n}\int_0^{u_n} t^{2j}(u_n - t)\,dt\right] h_n \\
&\quad + \left[\frac{2}{v_n - u_n}\left(\int_{u_n}^{(u_n+v_n)/2} t^{2j}(t - u_n)\,dt - \int_{(u_n+v_n)/2}^{v_n} t^{2j}(t - v_n)\,dt\right)\right] \ell_n \qquad (3.6) \\
&\quad + \left[\frac{2}{w_n - v_n}\left(\int_{v_n}^{(v_n+w_n)/2} t^{2j}(t - v_n)\,dt - \int_{(v_n+w_n)/2}^{w_n} t^{2j}(t - w_n)\,dt\right)\right] d_n = \delta_j,
\end{aligned}$$

$j = 0, 1, 2$, where $\delta_0 = \frac{1}{2}$, $\delta_1 = \delta_2 = 0$. Now let $u_n = o(1)$ and $w_n - u_n = o(u_n)$. Then $\{K_n^{(2)}(t)\}$ peaks, $K_n^{(2)}(t)$ decreases on $[0, \alpha_{n1}] = [0, u_n]$, and $v_n = O(u_n)$, $w_n = O(u_n)$. Solving (3.6), we obtain

$$d_n = \frac{4u_n^3(v_n - u_n)\, D_n}{u_n(v_n - u_n)(w_n - v_n)(w_n - u_n)\, D_n'} = \frac{4u_n^2 D_n}{(w_n - v_n)(w_n - u_n)\, D_n'}$$

where $D_n' \sim u_n D_n$. Hence,

$$\int_{v_n}^{a} K_n^{(2)}(t)\, dt = \tfrac{1}{2}(w_n - v_n)\, d_n \sim \frac{2u_n}{w_n - u_n} \to \infty \qquad \text{as} \quad n \to \infty,$$

which implies $\{M_0(| K_n^{(2)} |)\}$ is unbounded. Therefore, even though $\{K_n^{(2)}(t)\}$ peaks and $K_n^{(2)}(t)$ decreases on $[0, u_n] = [0, \alpha_{n1}]$ for each n, the associated sequence of approximating functions $\{\mathscr{L}_n'^{(2)}(f; x)\}$ does not converge for some $f \in C_a$.

In the construction of the above example, the assumption that $(w_n - u_n) = (w_n - \alpha_{n1}) = o(u_n)$ is necessary in order to show that $\{M_0(| K_n^{(2)} |)\}$ is not bounded. In particular, the condition $(v_n - u_n) = (\alpha_{n2} - \alpha_{n1}) = o(\alpha_{n1})$ is necessary (although not sufficient) for the argument. We see then that if the sequence of kernels $\{K_n^{(2)}\}$ constructed in the example peaks, and if α_{n2} does not get too *close* to α_{n1}, in the sense that

$$\overline{\lim_{n\to\infty}}\, \alpha_{n1}/\alpha_{n2} < 1, \tag{3.7}$$

then $\{\mathscr{L}_n^{(2)}(f; x)\}$ converges uniformly. We find, however, that condition (3.7) is still not sufficient to prove convergence for an arbitrary sequence of 4-zero operators. An example which shows this is given in [2].

The crux of the problem lies in the manner in which the set S of points between $[0, a]$ and the graph of $K_n^{(2)}(t)$ is distributed with respect to $[\alpha_{n1}, \alpha_{n2}]$. If $K_n^{(2)}(t)$ is such that most of the *mass* of S is sufficiently concentrated above $[\alpha_{n2}, a]$ as $n \to \infty$ the integrals $M_0(| K_n^{(2)} |)$ may become unbounded. This possibility can be eliminated by requiring $K_n^{(2)}(t)$ to satisfy certain monotonicity conditions on $[\alpha_{n1}, \alpha_{n2}]$ and on $[0, \alpha_{n1}]$.

The above remarks are generalized and made precise in the following definition and theorem.

Definition 4. Let $\{K_n^{(k)}(t)\}$ be a sequence of $2k$-zero kernels which peaks, $k \geqslant 2$. If there are $k - 1$ sequences of numbers $\{\beta_{ni}\}_{i=1}^{k-1}$ such that

(i) $\alpha_{ni} < \beta_{ni} < \alpha_{n,i+1}$,

(ii) each $| K_n^{(k)}(t)|$ decreases on each of the intervals $[0, \alpha_{n1}]$, $[\beta_{ni}, \alpha_{n,i+1}]$, $i = 1,\ldots, k - 1$,

(iii) $\overline{\lim_{n\to\infty}} \dfrac{\beta_{ni}}{\alpha_{n,i+1}} < 1, \qquad i = 1, 2,..., k-1,$

then the sequence $\{K_n^{(k)}\}$ is said to be *well-distributed.* (Note that condition (iii) implies

$$\overline{\lim_{n\to\infty}} \frac{\alpha_{ni}}{\alpha_{n,i+1}} < 1, \qquad i = 1,..., k-1).$$

THEOREM 3. *Let $\{\mathscr{L}_n^{(k)}\}$ be a sequence of 2k-zero operators which peaks, $k \geqslant 2$. If the associated sequence of kernels $\{K_n^{(k)}\}$ is well-distributed, then $\{\mathscr{L}(_n^{(k)} f; x)\}$ converges uniformly to $f \in C_a$.*

Proof. We first prove the theorem for the case $k = 2$. Let

$$A_n = \int_0^{\alpha_{n1}} K_n^{(2)}(t)\, dt, \qquad \text{and} \qquad D_n = \int_{\alpha_{n2}}^a K_n^{(2)}(t)\, dt.$$

We show, first, that A_n is bounded. Define a sequence of even functions $\{K_n(t)\}$ by

$$K_n(t) = \frac{\mu(\alpha_{n2}) - \mu(t)}{\mu(\alpha_{n2})} K_n^{(2)}(t), \qquad t \in (-a, a).$$

If $a \neq \infty$, extend $K_n(t)$ to the whole real line by continuity and periodicity with period 2a. Then $K_n(t) \in C_a$. Since $K_n^{(2)}(t)$ is a 4-zero kernel,

$$M_0(K_n) = M_0(K_n^{(2)}) - M_1(\mu, K_n^{(2)})/\mu(\alpha_{n2}) = 1$$

and

$$M_1(K_n) = M_1(\mu, K_n^{(2)}) - M_2(\mu, K_n^{(2)})/\mu(\alpha_{n2}) = 0.$$

Therefore, $\{K_n\}$ is a sequence of 2-zero kernels, and $\{K_n\}$ peaks since the only simple zeros of K_n are $\pm\alpha_{n1}$. Furthermore, since $\mu(\alpha_{n2}) - \mu(t)$ and $K_n^{(2)}(t)$ are positive decreasing functions on $[0, \alpha_{n1}]$, so is $K_n(t)$. Thus, the sequence of operators $\{\mathscr{L}_n\}$ defined by $\{K_n\}$ satisfies the hypothesis of Theorem 2, so that $\{\mathscr{L}_n\}$ converges uniformly. By Theorem 1b, $\{M_0(|K_n|)\}$ is bounded, say by M. Then

$$M \geqslant M_0(| K_n |) > \frac{1}{\mu(\alpha_{n2})} \int_0^{\alpha_{n1}} | \mu(\alpha_{n2}) - \mu(t)| \, | K_n^{(2)}(t)| \, dt$$

$$\geqslant \frac{\mu(\alpha_{n2}) - \mu(\alpha_{n1})}{\mu(\alpha_{n2})} \int_0^{\alpha_{n1}} | K_n^{(2)}(t)| \, dt,$$

i.e.,

$$M > \left(1 - \frac{\mu(\alpha_{n1})}{\mu(\alpha_{n2})}\right) A_n, \qquad \text{for all } n. \tag{3.8}$$

Since $\{K_n^{(2)}\}$ is well-distributed, there is a sequence $\{\beta_{n1}\}$ such that $\alpha_{n1} < \beta_{n1} < \alpha_{n2}$ and $\beta_{n1}/\alpha_{n2} \leqslant \eta < 1$. Hence, $\alpha_{n1}/\alpha_{n2} \leqslant \eta < 1$. By Fact 5, $\mu(\alpha_{n1})/\mu(\alpha_{n2}) \leqslant \eta' < 1$. Therefore, (3.8) yields

$$A_n < M/(1 - \eta'), \qquad \text{for all} \quad n. \tag{3.9}$$

Next, we show that D_n is bounded. Since $\{K_n^{(2)}\}$ is well-distributed, $|K_n^{(2)}(t)|$ decreases on $[\beta_{n1}, \alpha_{n2}]$. Hence, there is an $h_n > 0$ such that with

$$B_n = \int_{\alpha_{n1}}^{\beta_{n1}} |K_n^{(2)}(t)|\, dt \qquad \text{and} \qquad C_n = \int_{\beta_{n1}}^{\alpha_{n2}} |K_n^{(2)}(t)|\, dt$$

we have

$$h_n(\alpha_{n2} - \beta_{n1}) = C_n = A_n - B_n + D_n - \tfrac{1}{2}. \tag{3.10}$$

The situation is illustrated in Fig. 4.

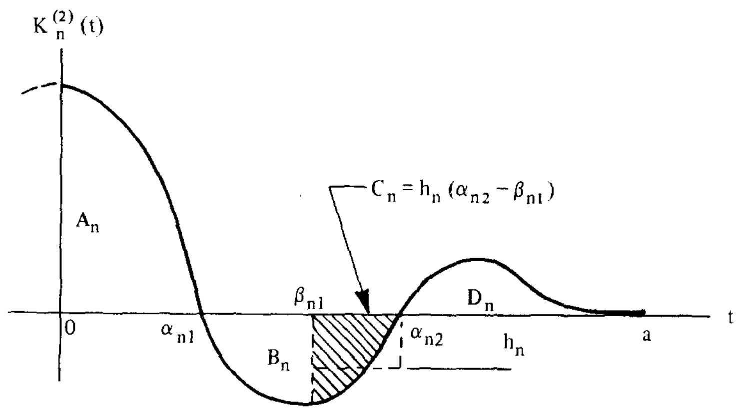

FIG. 4. 4-zero kernels.

By the same argument used in establishing inequality (3.3), we have

$$\int_{\beta_{n1}}^{\alpha_{n2}} \mu(t)\, K_n^{(2)}(t)\, dt < h_n \int_{\beta_{n1}}^{\alpha_{n2}} \mu(t)\, dt. \tag{3.11}$$

Since $\mu(t)$ is strictly increasing on $[0, a]$, there are positive numbers $c_n < 1$ such that

$$\int_{\beta_{n1}}^{\alpha_{n2}} \mu(t)\, dt = c_n(\alpha_{n2} - \beta_{n1})\, \mu(\alpha_{n2}). \tag{3.12}$$

Therefore, from (3.10), (3.11) and (3.12), we have

$$0 = \int_0^a \mu(t)\, K_n^{(2)}(t)\, dt = \int_0^{\alpha_{n1}} \mu(t)\, K_n^{(2)}(t)\, dt - \int_{\alpha_{n1}}^{\beta_{n1}} \mu(t) \mid K_n^{(2)}(t)| \, dt$$

$$- \int_{\beta_{n1}}^{\alpha_{n2}} \mu(t) \mid K_n^{(2)}(t)| \, dt + \int_{\alpha_{n2}}^{a} \mu(t)\, K_n^{(2)}(t)\, dt$$

$$> - \mu(\beta_{n1})\, B_n - c_n h_n(\alpha_{n2} - \beta_{n1})\, \mu(\alpha_{n2}) + \mu(\alpha_{n2})\, D_n$$

$$= [c_n \mu(\alpha_{n2}) - \mu(\beta_{n1})]\, B_n - [c_n \mu(\alpha_{n2})]\, A_n$$

$$+ [(1 - c_n)\, \mu(\alpha_{n2})]\, D_n + \tfrac{1}{2} c_n \mu(\alpha_{n2}). \tag{3.13}$$

Divide this inequality through by $\mu(\alpha_{n2}) > 0$ to obtain

$$B_n \left(c_n - \frac{\mu(\beta_{n1})}{\mu(\alpha_{n2})} \right) + D_n(1 - c_n) < c_n(A_n - \tfrac{1}{2}). \tag{3.14}$$

From the definition of c_n, we have $c_n(\alpha_{n2} - \beta_{n1})\, \mu(\alpha_{n2}) = (\alpha_{n2} - \beta_{n1})\, \mu(\xi)$, where $\beta_{n1} < \xi < \alpha_{n2}$; hence

$$c_n = \mu(\xi)/\mu(\alpha_{n2}) > \mu(\beta_{n1})/\mu(\alpha_{n2}).$$

The coefficient of B_n in (3.14) is therefore positive, so that we have

$$0 < D_n < \frac{c_n(A_n - \frac{1}{2})}{(1 - c_n)}.$$

By Fact 6, there exists a $\delta > 0$ such that $c_n \leqslant \delta < 1$, and from (3.9), A_n is bounded. Hence, so is D_n. It follows that $\{M_0(| K_n^{(2)} |)\}$ is bounded. By Theorem 1a, then, $\{\mathscr{L}_n^{(2)}(f; x)\}$ converges uniformly to $f(x) \in C_a$.

We now proceed by induction on k. Assume the theorem is true for $k = m - 1$. We show that it is true for $k = m$. Let $\{K_n^{(m)}\}$ be a sequence of well-distributed $2m$-zero kernels which peaks. Define a sequence of even functions $\{K_n(t)\}$ by

$$K_n(t) = \frac{\mu(\alpha_{nm}) - \mu(t)}{\mu(\alpha_{nm})}\, K_n^{(m)}(t), \qquad t \in (-a, a).$$

If $a \neq \infty$, extend $K_n(t)$ continuously and periodically to the whole real line, so that $K_n \in C_a$. By a similar argument as was used in the 4-zero case, we can conclude that $\{K_n\}$ is a well-distributed sequence of $(2m - 2)$-zero kernels which peaks, and the only positive simple zeros of $K_n(t)$ are the first $m - 1$ positive simple zeros of $K_n^{(m)}(t)$. Thus, the sequence of operators $\{\mathscr{L}_n\}$ defined by $\{K_n\}$ satisfies the induction hypotheses, and therefore $\{\mathscr{L}_n(f; x)\}$ converges uniformly to $f(x)$. By Theorem 1b, $\{M_0(| K_n |)\}$ is bounded, say by M. Define

$$A_{ni} = \int_{\alpha_{n,i-1}}^{\alpha_{ni}} | K_n^{(m)}(t)|, \qquad i = 1, \ldots, m - 1, \quad \alpha_{n0} = 0.$$

Then

$$M \geqslant M_0(|K_n|) = \frac{1}{\mu(\alpha_{nm})} \int_{-a}^{a} |\mu(\alpha_{nm}) - \mu(t)| \, |K_n^{(m)}(t)| \, dt$$
$$> \frac{1}{\mu(\alpha_{nm})} \int_{\alpha_{n,i-1}}^{\alpha_{ni}} |\mu(\alpha_{nm}) - \mu(t)| \, |K_n^{(m)}(t)| \, dt$$
$$> \frac{\mu(\alpha_{nm}) - \mu(\alpha_{ni})}{\mu(\alpha_{nm})} A_{ni}, \qquad i = 1,..., m-1. \tag{3.15}$$

Since $\{K_n^{(m)}\}$ is well-distributed, there exist sequences $\{\beta_{ni}\}$ such that $\alpha_{ni} < \beta_{ni} < \alpha_{n,i+1}$ and $\beta_{ni}/\alpha_{nm} < \beta_{ni}/\alpha_{n,i+1} \leqslant \eta < 1$. Hence, $\alpha_{ni}/\alpha_{nm} \leqslant \eta < 1$. By Fact 5, for all large n, $\mu(\alpha_{ni})/\mu(\alpha_{nm}) \leqslant \eta' < 1$. Therefore, (3.15) yields

$$A_{ni} < M/(1-\eta'), \qquad i = 1,..., m-1. \tag{3.16}$$

Now define

$$B_n = \int_{\alpha_{n,m-1}}^{\beta_{n,m-1}} |K_n^{(m)}(t)| \, dt, \qquad C_n = \int_{\beta_{n,m-1}}^{\alpha_{nm}} |K_n^{(m)}(t)| \, dt,$$
$$D_n = \int_{\alpha_{nm}}^{a} |K_n^{(m)}(t)| \, dt.$$

We show that D_n is bounded. Assume m is even. Since $|K_n(t)|$ decreases on $[\beta_{n,m-1}, \alpha_{nm}]$, there is an $h_n > 0$ such that

$$C_n = h_n(\alpha_{nm} - \beta_{n,m-1}) = \sum_{i=1}^{m-1} (-1)^{i+1} A_{ni} - B_n + D_n - \tfrac{1}{2} \tag{3.17}$$

and

$$\int_{\beta_{n,m-1}}^{\alpha_{nm}} \mu(t) \, |K_n(t)| \, dt < h_n \int_{\beta_{n,m-1}}^{\alpha_{nm}} \mu(t) \, dt = h_n c_n (\alpha_{nm} - \beta_{n,m-1}) \, \mu(\alpha_{nm})$$

where $0 < c_n < 1$. Therefore, if we apply the same procedure used in deriving (3.13) in the case $k = 2$, we obtain

$$0 = \int_0^a \mu(t) \, K_n(t) \, dt > \sum_{i=1}^{(m-2)/2} \mu(\alpha_{n,2i})(A_{n,2i+1} - A_{n,2i})$$
$$- \mu(\beta_{n,m-1}) \, B_n - c_n \mu(\alpha_{nm}) \, C_n + \mu(\alpha_{nm}) \, D_n \, .$$

If we now substitute the expression for C_n in (3.17) into the above relation, divide through by $\mu(\alpha_{nm})$, delete all positive terms (except the D_n term) from the right side of the resultant inequality, and solve for D_n, we obtain

$$0 < D_n < \frac{1}{1 - c_n} \left(\sum_{i=1}^{(m-2)/2} \frac{\mu(\alpha_{n,2i})}{\mu(\alpha_{nm})} A_{n,2i} + c_n \sum_{i=1}^{m/2} A_{n,2i-1} \right).$$

By Fact 6, $c_n \leqslant \delta < 1$ for some δ, and from (3.16), all A_{ni} are bounded, $i = 1,..., m - 1$. Hence, D_n is bounded. A similar argument can be used to show that D_n is bounded for m odd. This, together with (3.16), implies $\{M_0(| K_n^{(m)} |)\}$ is bounded and, hence, $\{\mathscr{L}_n^{(m)}(f; x)\}$ converges uniformly to $f \in C_a$. This completes the induction step and the theorem is established.

Remarks

In the definition of a sequence of $2k$-zero kernels we required that $M_0(K_n^{(k)}) = 1$ and $M_j(\mu, K_n^{(k)}) = 0, j = 1,..., k$, for each n. These conditions, however, may be slightly weakened by requiring only that

$$\lim_{n\to\infty} M_j(\mu, K_n^{(k)}) = \delta_j, \qquad j = 0, 1,..., k,$$

where $\delta_0 = 1$, $\delta_j = 0, j \geqslant 1$, and that, for at least one $j \geqslant 1$, $M_j(\mu, K_n^{(k)}) \geqslant 0$ for all n. All convergence results still hold under this less restrictive definition.

The continuity requirement on $K_n^{(k)}$ may also be relaxed to include kernels which have a finite number of jump discontinuities.

Both these modifications can easily be incorporated into the proof of each theorem.

A discussion on degree of convergence for special classes of $2k$-zero operators and a method for constructing $2k$-zero operators will be presented in a subsequent article.

References

1. P. L. Butzer, Representation and approximation of functions by general singular integrals, IA, IB, *Nederl. Akad. Wetensch. Proc. Ser. A* **63** (1960), 1–24.
2. J. C. Hoff, "Approximation with Kernels of Finite Oscillations," Thesis, Purdue University, Lafayette, Indiana, 1968.
3. P. P. Korovkin, Linear operators and approximation theory, Hindustan Publishing Corp., Delhi, India, 1959.
4. G. G. Lorentz, "Approximation of Functions," Holt, Rinehart and Winston, New York, 1966.
5. F. Schurer, On linear positive operators in approximation theory, Thesis, Technological University, Delft, Netherlands, 1965.

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